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Differential geometric mechanisms in Ostrohrads'kyj relativistic spherical top dynamics

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Some intrinsic tools from the formal theory of variational equations are being demonstrated at work in application to one concrete example of the third-order evolution equation of free relativistic top in three-dimensional space-time. The main goal is to introduce a combined approach consisting in the simultaneous utilization of symmetry principles along with the inverse variational problem considerations in terms of vector-valued differential forms. Next, some simple algorithm of transition between the autonomous variational problem and the variational problem in parametric form is established. The example definitely solved shows no-existence of a globally and intrinsically defined Lagrangian for the Poincaré-invariant and well defined unique variational equation in the case in hand. Hamiltonian counterpart is briefly discussed in terms of Poisson bracket. The model appears to provide a generalized canonical description of the quasi-classical spinning particle governed by the Mathisson-Papapetrou equations in flat space-time.

Introduction

Ostrohrads'kyj's mechanics has been repeatedly revisited from the point of view of global analysis including certain features of intrinsic differential geometry (see monographs [1–3], preceded and followed by large number of other reviews and articles). The differential geometry of Ostrohrads'kyj's mechanics gained a renewed interest on the part of contemporary mathematicians during past two decades.

On the other hand, applications of the higher-order variational calculus to some classical models of a relativistic particle motion began in 1937 and continue till now. The investigations on the application of Ostrohrads'kyj's mechanics to real physical models haven't been abandoned since the pioneer works by Mathisson, Bopp, Weyssenhoff, Raabe, and Hönl (see [4–7]). Most of the applications consider models of test particles endowed with inner degrees of freedom [8–15] or models which put the notion of the acceleration onto the framework of general differential geometric structure of the ex-

tended configuration space of the particle [16]. One interesting example of how the derivatives of the third order appear in the equations of motion of test particle is provided by Mathisson-Papapetrou equations

$$\begin{aligned} \frac{D}{d\zeta} \left(m_0 \frac{u^\alpha}{\|\mathbf{u}\|} + \frac{u_\gamma}{\|\mathbf{u}\|^2} \frac{D}{d\zeta} S^{\alpha\gamma} \right) &= \mathcal{F}^\alpha \\ \frac{D}{d\zeta} S^{\alpha\beta} &= \frac{1}{\|\mathbf{u}\|^2} \left(u^\beta u_\gamma \frac{D}{d\zeta} S^{\alpha\gamma} - u^\alpha u_\gamma \frac{D}{d\zeta} S^{\beta\gamma} \right) \end{aligned} \quad (1)$$

together with the supplementary condition

$$u_\gamma S^{\alpha\gamma} = 0. \quad (3)$$

It is immediately clear that the second term in (1) may produce the derivatives of the third order of space-time variables x^α as soon as one dares to substitute $u_\gamma \frac{DS^{\alpha\gamma}}{d\zeta}$ by $-S^{\alpha\gamma} \frac{Du^\gamma}{d\zeta}$ in virtue of (3). Such substitution in fact means differentiating equation (3). However, the system of equations thus obtained will not possess any additional solutions comparing to that of (1–3) as far as one does not forget of the original constraint (3). The system (1–3) was recently a subject of discussion in [17]. In (1) the right hand side vanishes if there is no gravitation.

It is a matter of common consent that the relativistic motion of simple particles in gravitational field may be described mathematically via the notion of geodesic paths. Because less simple particles obey higher-order equations of motion, it seems worthwhile to investigate the appropriate geometries. But, in the same way as pseudo-Riemannian geometry descends down to the natural representation of Lorentz group, more complicated geometry should break out first from some symmetry considerations of global character.

We intend to present in this contribution some tools from the arsenal of intrinsic analysis on manifolds that may appear helpful in solving the invariant inverse problem of the calculus of variations. In special case of three-dimensional space-time we shall successfully follow some prescriptions for obtaining a third-order Poincaré-invariant variational equations up to the very final solution thus discovering the unique possible one, which will then be identified with the motion of free relativistic top by means of comparing it to (1–3) when $R^\alpha_{\beta\delta\gamma} = 0$. This case of two-dimensional motion in space makes quite a good sense from the viewpoint of the general theory as well [18]. On the other hand, one can show directly that even in four-dimensional special relativity case the world line of a particle obeying the system of equations (1–3) has the third curvature equal to zero (see also [19]). Thus, even in this case the particle actually propagates in two-dimensional space. Another feature of this limited case is that the spin four-vector

$$\sigma_\alpha = \frac{1}{2\|\mathbf{u}\|} \epsilon_{\alpha\beta\gamma\delta} u^\beta S^{\gamma\delta} \quad (4)$$

keeps constant under the condition of the motion be free. So knowing a Lagrange function for some third-order equation, equivalent to (1–3), allows offering a generalized Hamiltonian description in terms of Poisson bracket that might be considered as a canonical equivalent to (1, 3). Our example exposes some typical features of variational calculus:

- the nonexistence (in our case) of well defined invariant Lagrangian along with intrinsically very well defined equation of motion with Poincaré symmetry produced by each of a family of degenerate Lagrangians which transform into each other by renumbering the axes of Lorentz frame;
- all handled Lagrangians give rise to the same system of canonical equations;
- each Lagrangian includes different set of second order derivatives, thus their sum is not a Lagrangian of minimal order.

1 Homogeneous form and parametric invariance

Presentation of the equation of motion in so-called ‘manifestly covariant form’ stipulates introducing of the space of Ehresmann’s velocities of the configuration manifold M of the particle, $T^k M = \{x^\alpha, \dot{x}^\alpha, \ddot{x}^\alpha \dots x^\alpha_{(k)}\}$. In future the notations u^α , \dot{u}^α , \ddot{u}^α , $u^\alpha_{(r)}$ will frequently be used in place of \dot{x}^α , \ddot{x}^α , $x^\alpha_{(3)}$, $x^\alpha_{(r+1)}$, and also $x^\alpha_{(0)}$ sometimes will merely denote the x^α . We call some mapping $\zeta \mapsto x^\alpha(\zeta)$ the *parametrized* (by means of ζ) *world line* and its image in M will be called the *non-parameterized world line*. As far as we are interested in a variational equation (of order s) that would describe the non-parameterized world lines of the particle,

$$\mathcal{E}_\alpha \left(x^\alpha, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha, \dots, u^\alpha_{(s-1)} \right) = 0, \quad (5)$$

the Lagrange function \mathcal{L} has to satisfy the Zermelo conditions, which in our case of only up to the second order derivatives present in \mathcal{L} read

$$\begin{aligned} u^\beta \frac{\partial}{\partial u^\beta} \mathcal{L} + 2\dot{u}^\beta \frac{\partial}{\partial \dot{u}^\beta} \mathcal{L} &= \mathcal{L} \\ u^\beta \frac{\partial}{\partial \dot{u}^\beta} \mathcal{L} &= 0. \end{aligned}$$

In this approach the independent variable ζ (called the *parameter along the world line*) is not included into the configuration manifold M . Thus the space $T^k M$ is the appropriate candidate for the role of the underlying manifold on where the variational problem in the autonomous form should be posed. We may include the parameter ζ into the configuration manifold by introducing the trivial fibre manifold $\mathbb{R} \times M \rightarrow \mathbb{R}$, $\zeta \in \mathbb{R}$, and putting into consideration its k^{th} -order prolongation, $J^k(\mathbb{R}, M)$, i. e. the space, constituted by the k^{th} -order jets of local cross-sections of $Y = \mathbb{R} \times M$ over \mathbb{R} . Each such cross-section of Y is nothing but the graph in $\mathbb{R} \times M$ of some local curve $x^\alpha(\zeta)$ in M . For each $r \in \mathbb{N}$ there exists an obvious projection

$$p_0^r : J^r(\mathbb{R}, M) \rightarrow T^r M \quad (6)$$

as follows. The manifold $T^r M$ consists of the derivatives up to the r^{th} -order of curves $x^\alpha(\zeta)$ in M , evaluated at $0 \in \mathbb{R}$. If for every $\tau \in \mathbb{R}$ we denote by same character τ the mapping $\zeta \mapsto \zeta + \tau$ of \mathbb{R}

onto itself, then the projection reads

$$p_0^r : \left(\tau; x^\alpha(\tau), \frac{d}{d\zeta} x^\alpha(\tau), \frac{d^2}{d\zeta^2} x^\alpha(\tau), \dots, \dots, \frac{d^r}{d\zeta^r} x^\alpha(\tau) \right) \mapsto \left((x^\alpha \circ \tau)(0), \frac{d}{d\zeta} (x^\alpha \circ \tau)(0), \frac{d^2}{d\zeta^2} (x^\alpha \circ \tau)(0), \dots, \frac{d^r}{d\zeta^r} (x^\alpha \circ \tau)(0) \right) \quad (7)$$

By means of the projection (6, 7) every Lagrange function \mathcal{L} initially defined on $T^k M$ may be pulled back to the manifold $J^k(\mathbb{R}, M)$ and defines there some function \mathcal{L}_0 by the obvious formula $\mathcal{L}_0 = \mathcal{L} \circ p_0^k$. We say that the differential form

$$\lambda = \mathcal{L}_0 d\zeta \quad (8)$$

constitutes a variational problem in extended parametric form because in the construction of the new configuration manifold $\mathbb{R} \times M$ the independent variable ζ was artificially doubled. But we shall need this construction later.

Let us return to the variational problem set on the manifold $T^k M$ by a given Lagrange function \mathcal{L} . The very first moment we impose the Zermelo conditions, the problem becomes degenerate. There exists one way to avoid degeneracy by reducing the number of velocities. Of course, at the cost of losing the “homogeneity” property of the equation (5). Consider some way of segregating the variables $x^\alpha \in M$ into $t \in \mathbb{R}$ and $x^i \in Q$, $\dim Q = \dim M - 1$, thus making M into some fibration, $M \approx \mathbb{R} \times Q$, over \mathbb{R} . The manifold of jets $J^r(\mathbb{R}, Q)$ provides some local representation of what is known as the manifold $C^r(M, 1)$ of r -contact one-dimensional submanifolds of M . Intrinsically defined global projection of non-zero elements of $T^r M$ onto the manifold $C^r(M, 1)$ in this local and, surely, “non-covariant” representation is given by

$$\wp^r : T^r M \setminus \{0\} \rightarrow J^r(\mathbb{R}, Q), \quad (9)$$

and in the third order is implicitly defined by the following formulae, where the local coordinates in $J^r(\mathbb{R}, Q)$ are denoted by $t; x^i, v^i, v'^i, v''^i, \dots, v_{(r-1)}^i$

with $v_{(0)}^i$ marking v^i sometimes,

$$\begin{aligned} \dot{t} v^i &= u^i \\ (t)^3 v'^i &= \dot{t} u^i - \ddot{t} u^i \\ (t)^5 v''^i &= (t)^2 \ddot{u}^i - 3 \dot{t} \ddot{u}^i + [3(\dot{t})^2 - \ddot{t} t_{(3)}] u^i. \end{aligned} \quad (10)$$

There does not exist any well-defined projection from the manifold $C^r(M, 1)$ onto the space of independent variable \mathbb{R} , so the expression

$$\Lambda = L \left(t; x^i, v^i, v'^i, v''^i, \dots, v_{(k-1)}^i \right) dt \quad (11)$$

will vary in the dependence on the concrete way of local representation $M \approx \mathbb{R} \times Q$. We say that two different expressions of type (11) define one and the same variational problem in parametric form if their difference expands into nothing but only the pull-backs to $C^k(M, 1)$ of the following contact forms, which live on the manifold $C^1(M, 1)$,

$$\theta^i = dx^i - v^i dt. \quad (12)$$

These differential forms obviously vanish along the jet of every curve $\mathbb{R} \rightarrow Q$.

Let the components of the variational equation

$$E_i = 0 \quad (13)$$

of the Lagrangian (11) be treated as the components of the following vector one-form,

$$\mathbf{e} = \{E_i dt\}. \quad (14)$$

We intend to give a “homogeneous” description to (14) and (11) in terms of some objects that would live on $T^s M$ and $T^k M$ respectively. But we cannot apply directly the pull-back operation to the Lagrangian (11) because the pull-back of one-form is a one-form again, and what we need on $T^k M$ is a Lagrange *function*, not a differential form. However, it is possible to pull (11) all the way back along the composition of projections (6) and (9),

$$pr^k = \wp^k \circ p_0^k, \quad (15)$$

ultimately to the manifold $J^k(\mathbb{R}, M)$. In such a way we obtain the differential form $(L \circ pr^k) dt$. But what we do desire, is a form that should involve $d\zeta$ solely (i. e. semi-basic with respect to the projection $J^k(\mathbb{R}, M) \rightarrow \mathbb{R}$). Fortunately, the two

differential forms, dt and $\dot{t}d\zeta$, differ not more than only by the contact form

$$\vartheta = dt - \dot{t}d\zeta. \quad (16)$$

Now, we recall that equivalent Lagrangians that have the structure of (11) differ by multiplies of the contact forms (12). It remains to notice that, by the course of (7) and (10), the pull-backs of the contact forms (12) expand only into contact forms (16) and

$$\vartheta^i = dx^i - u^i d\zeta \quad (17)$$

alone,

$$pr^{1*}\vartheta^i = dx^i - (v^i \circ pr^1)dt = \vartheta^i - (v^i \circ pr^1)\vartheta.$$

Thus, every variational problem, posed on $J^k(\mathbb{R}, Q)$ and represented by (11), transforms into an equivalent variational problem

$$\lambda = (L \circ pr^k)\dot{t}d\zeta, \quad (18)$$

posed on $J^k(\mathbb{R}, M)$. But the Lagrange function of this new variational problem,

$$\mathcal{L}_0 = (L \circ pr^k)\dot{t}, \quad (19)$$

does not depend upon the parameter ζ and substantially may be thought of as a function, defined on $T^k M$.

We prefer to cast the variational equation (of some order $s \leq 2k$), generated by the Lagrangian (18), into the framework of vector-valued exterior differential systems theory by introducing the following vector differential one-form, defined on the manifold $J^s(\mathbb{R}, M)$,

$$\varepsilon = \mathcal{E}_\alpha \left(x^\alpha, \dot{x}^\alpha, \dots, x_{(s)}^\alpha \right) d\zeta. \quad (20)$$

The expressions $\mathcal{E}_\alpha \left(x^\alpha, \dot{x}^\alpha, \dots, x_{(s)}^\alpha \right)$ in (20) may also be treated as ones, defined on $T^s M$, similar to \mathcal{L}_0 . Altogether the constructions, built above, allow formulation of the following statement:

Proposition 1 *If the differential form (14) corresponds to the variational equation of the Lagrangian (11), then the expressions*

$$\mathcal{E}_\alpha = \{ -u^i E_i, \dot{t} E_i \} \quad (21)$$

correspond to the Lagrange function (19).

In this case the (s^{th} -order) equation (5) describes “in homogeneous form” the same non-parameterized world lines of a particle governed by the variational problem (19), as does the equation (13) with the Lagrangian given by (11), and also \mathcal{L}_0 obviously satisfies the Zermelo conditions. As to more sophisticated details, paper [20] may be consulted.

2 Criterion of variability

Our main intention is to find a Poincaré-invariant ordinary (co-vector) differential equation of the third order in three-dimensional space-time. With this goal in mind we organize the expressions E_i in (14) into a single differential object, the exterior one-form

$$\mathbf{e}_0 = E_i dx^i \quad (22)$$

defined on the manifold $J^s(\mathbb{R}, Q)$, so that the vector differential form (14) should now be treated as the coordinate representation of the intrinsic differential geometric object

$$\mathbf{e} = e_i dx^i = E_i dt \otimes dx^i = dt \otimes \mathbf{e}_0. \quad (23)$$

This way constructed differential form \mathbf{e} is an element of the graded module of differential semi-basic with respect to \mathbb{R} differential forms on $J^s(\mathbb{R}, Q)$ with values in the bundle of graded algebras $\wedge T^*Q$ of scalar forms on TQ . Of course, due to the dimension of \mathbb{R} , actually only functions (i. e. semi-basic zero-forms) and semi-basic one-forms (i. e. in dt solely) exist. We also wish to mention that every (scalar) differential form on Q is naturally treated as a differential form on $T^r Q$, i. e. as an element of the graded algebra of cross-sections of $\wedge T^*(T^r Q)$.

For arbitrary $s \in \mathbb{N}$ let $\Omega_s(Q)$ denote the algebra of (scalar) differential forms on $T^s Q$ with coefficients depending, aside of v_{r-1} , $r \leq s$, also on $t \in \mathbb{R}$. It is possible to develop some calculus in $\Omega_s(Q)$ by introducing the operator of vertical (with respect to \mathbb{R}) differential d_v and the operator of total (or formal “time”) derivative D_t by means of the prescriptions:

$$d_v f = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial v_{(r)}^i} dv_{(r)}^i, \quad d_v^2 = 0,$$

so that $d_v x^i$ and $d_v v_{(r)}^i$ coincide with dx^i and $dv_{(r)}^i$ respectively, and

$$D_t f = \frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + v_{(r+1)}^i \frac{\partial f}{\partial v_{(r)}^i}, \quad D_t d_v = d_v D_t.$$

There exists a notion of *derivation* in graded algebras endowed with generalized commutation rule, as $\Omega_s(Q)$ is. An operator D is called a derivation of degree q if for any differential form ϖ of degree p and any other differential form w it is true that $D(\varpi \wedge w) = D(\varpi) \wedge w + (-1)^{pq} \varpi \wedge D(w)$. To complete the above definitions it is necessary to demand that d_v be a derivation of degree 1 whereas D_t be a derivation of degree 0. But still this is not the whole story. We need one more derivation of degree 0, denoted here as ι , and defined by its action on functions and one-forms, which altogether locally generate the algebra $\Omega_s(Q)$,

$$\iota f = 0, \iota dx^i = 0, \iota dv^i = dx^i, \iota dv_{(r)}^i = (r+1) dv_{(r-1)}^i.$$

Let operator \deg mean evaluating of the degree of a differential form. The *Lagrange differential* δ is first introduced by its action upon the elements of $\Omega_s(Q)$,

$$\delta = \left(\deg + \sum_{r=1}^s \frac{(-1)^r}{r!} D_t^r \iota^r \right) d_v,$$

and next trivially extended to the whole of the graded module of semi-basic differential forms on $J^s(\mathbb{R}, Q)$ with values in $\wedge T^*(T^r Q)$ by means of

$$\begin{aligned} \delta(\omega_i dt \otimes dx^i) &= dt \otimes \delta(\omega_i dx^i), \\ \delta(\omega_i^r dt \otimes dv_{(r)}^i) &= dt \otimes \delta(\omega_i^r dv_{(r)}^i). \end{aligned}$$

This δ turns out to possess the property $\delta^2 = 0$. We have, that for the differential geometric objects (23) and (11) the following relation holds:

$$e = \delta \Lambda = dt \otimes \delta L. \quad (24)$$

Now the criterion for an arbitrary set of expressions $\{E_i\}$ in (14) be the variational equations for some Lagrangian reads

$$\delta e = dt \otimes \delta e_0 = 0, \quad (25)$$

with e constructed from $\{E_i\}$ by means of (22) and (23).

Of course, one may apply the above constructions literally to the analogous objects living on the manifold $J^s(\mathbb{R}, M)$ in (6) and obtain the operator, the Lagrange differential, δ^Y acting upon semi-basic, with respect to \mathbb{R} , differential forms on $J^s(\mathbb{R}, M)$ with values in the bundle $\wedge T^*(T^s M)$. In the algebra $\Omega_s(M)$ operator δ^Y preserves the sub-algebra of forms that do not depend on the parameter $\zeta \in \mathbb{R}$. The restriction of δ^Y to the algebra of differential forms truly defined on $T^s M$ sole will be denoted by δ^T . It was introduced in [21]. If in (8) the Lagrange function \mathcal{L}_0 does not depend on the parameter $\zeta \in \mathbb{R}$, as is the case of (18–19), rather than applying δ^Y to the forms λ from (8) and

$$\varepsilon = \varepsilon_\alpha dx^\alpha = \mathcal{E}_\alpha d\zeta \otimes dx^\alpha \quad (26)$$

from (20), we may apply the restricted operator δ^T to the Lagrange function \mathcal{L}_0 and to the differential form

$$\varepsilon_0 = \mathcal{E}_\alpha dx^\alpha. \quad (27)$$

In case of (19) the criteria $\delta^Y \varepsilon = 0$,

$$\delta^T \varepsilon_0 = 0, \quad (28)$$

and (25) are all equivalent, and the variational equations, produced by the expressions $\varepsilon = \delta^Y \lambda$ from (26, 18), $\varepsilon_0 = \delta^T \mathcal{L}_0$ from (27, 19), and e from (24) all are equivalent to (5). The expressions (14) and (11) are not “generally covariant” whereas (27) is. But the criterion (28) needs to be solved along with Zermelo conditions, whereas (25) is self-contained.

The presentation of a system of variational expressions $\{E_i\}$ under the guise of a semi-basic (i. e. in dt solely) differential form that takes values in the bundle of one-forms over the configuration manifold Q is quite natural:

- the Lagrange density (called *Lagrangian* in this work) is a one-form in dt only;
- the destination of the Euler-Lagrange expressions in fact consists in evaluating them on the infinitesimal variations, i. e. the vector fields tangent to the configuration manifold Q along the critical curve; consequently, the set

of E_i constitutes a linear form on the cross-sections of TQ with the coefficients depending on higher derivatives

More details can be found in [22] and [23].

3 Lepagean equivalent

The system of partial differential equations, imposed on E_i , that arises from (25) takes more tangible shape in the concrete case of third-order Euler-Poisson (i. e. ordinary Euler-Lagrange) expressions. The reader may consult [24] and references therein. Let skew-symmetric matrix \mathbf{A} , symmetric matrix \mathbf{B} , and a column \mathbf{c} all depend on t , x^i , and v^i and satisfy the following system of partial differential equations,

$$\begin{aligned} \partial_{v^i} A_{jl} &= 0, \\ 2B_{[ij]} - 3D_1 A_{ij} &= 0, \\ 2\partial_{v^i} B_{jl} - 4\partial_{x^i} A_{jl} + \partial_{x^l} A_{ij} + \\ &+ 2D_1 \partial_{v^l} A_{ij} = 0, \\ \partial_{v^i} c_j - D_1 B_{(ij)} &= 0, \\ 2\partial_{v^l} \partial_{v^i} c_j - 4\partial_{x^i} B_{jl} + D_1^2 \partial_{v^l} A_{ij} + \\ &+ 6D_1 \partial_{x^i} A_{jl} = 0, \\ 4\partial_{x^i} c_j - 2D_1 \partial_{v^i} c_j - D_1^3 A_{ij} &= 0, \end{aligned} \quad (29)$$

where the differential operator D_1 is the lowest order generator of the Cartan distribution,

$$D_1 = \partial_t + \mathbf{v} \cdot \partial_x.$$

It is obvious and commonly well known that the Euler-Lagrange expressions are of affine type in the highest derivatives. The most general form of the Euler-Poisson equation of the third order reads:

$$\mathbf{A} \cdot \mathbf{v}'' + (\mathbf{v}' \cdot \partial_v) \mathbf{A} \cdot \mathbf{v}' + \mathbf{B} \cdot \mathbf{v}' + \mathbf{c} = 0. \quad (30)$$

Due to the affine structure of the left hand side of equation (30), we may alongside with the differential form (23) introduce next one, the coefficients of which do not depend on third-order derivatives,

$$\begin{aligned} \epsilon &= A_{ij} dv'^j \otimes dx^i + k_i dt \otimes dx^i, \\ \mathbf{k} &= (\mathbf{v}' \cdot \partial_v) \mathbf{A} \cdot \mathbf{v}' + \mathbf{B} \cdot \mathbf{v}' + \mathbf{c}. \end{aligned} \quad (31)$$

From the point of view of searching only holonomic local curves in $J^3(\mathbb{R}, Q)$ those exterior differential systems who differ not more than merely by multipliers of the contact forms (12) and

$$\theta'^i = dv^i - v'^i dt, \quad \theta''^i = dv'^i - v''^i dt,$$

are considered equivalent. The differential forms (31) and (23) are equivalent:

$$\epsilon - e = A_{ij} \theta''^j \otimes dx^i.$$

The differential form (31) may be accepted as an alternative representation of the *Lepagean equivalent* [1] of (23).

4 Invariant Euler-Poisson equation

We are preferably interested in those variational equations that expose some symmetry. Let $X(\epsilon)$ denote the component-wise action of an infinitesimal generator X on a vector differential form ϵ . That the exterior differential system, generated by the form ϵ , possesses the symmetry of X means that there exist some matrices Φ , Ξ , and Π depending on \mathbf{v} and \mathbf{v}' , such that

$$X(\epsilon) = \Phi \cdot \epsilon + \Xi \cdot (\mathbf{x} - \mathbf{v} dt) + \Pi \cdot (d\mathbf{v} - \mathbf{v}' dt). \quad (32)$$

Equation (32) expresses the condition that two vector exterior differential systems, the one, generated by the vector differential form ϵ , and the other, generated by the shifted form $X(\epsilon)$, are algebraically equivalent. For systems, generated by one-forms (as in our case) this is completely the same thing as to demand that the set of local solutions be preserved under the one-parametric Lie subgroup generated by X . We see two advantages of this method:

- the symmetry conception is formulated in reasonably most general form;
- the problem of invariance of a differential equation is reformulated in algebraic terms by means of undetermined coefficients Φ , Ξ , and Π ;

- the order of the underlying non-linear manifold is reduced (to $J^2(\mathbb{R}, Q)$ instead of $J^3(\mathbb{R}, Q)$).

Further details may be found in [25].

In the case of the Poincaré group we assert that \mathbf{A} and \mathbf{k} in (31) do not depend upon t and \mathbf{x} . And for the sake of reference it is worthwhile to put down the general expression of the generator of the Lorentz group, parameterized by a skew-symmetric matrix Ω and some vector π :

$$\begin{aligned} X = & -(\pi \cdot \mathbf{x}) \partial_t + g_{00} t \pi \cdot \partial_{\mathbf{x}} + \Omega \cdot (\mathbf{x} \wedge \partial_{\mathbf{x}}) + \\ & + g_{00} \pi \cdot \partial_{\mathbf{v}} + (\pi \cdot \mathbf{v}) \mathbf{v} \cdot \partial_{\mathbf{v}} + \Omega \cdot (\mathbf{v} \wedge \partial_{\mathbf{v}}) + \\ & + 2(\pi \cdot \mathbf{v}) \mathbf{v}' \cdot \partial_{\mathbf{v}'} + (\pi \cdot \mathbf{v}') \mathbf{v} \cdot \partial_{\mathbf{v}'} + \\ & + \Omega \cdot (\mathbf{v}' \wedge \partial_{\mathbf{v}'}). \end{aligned}$$

Here the centred dot symbol denotes the inner product of vectors or tensors and the lowered dot symbol denotes the contraction of a row-vector and the subsequent column-vector.

System of equations (29, 32) may possess many solutions. Or no solutions at all, depending on the dimension of the configuration manifold. For example, in dimension one the skew-symmetric matrix \mathbf{A} does not exist. If $\dim Q = 3$, there is no solution to the P.D.E. system (29, 32) (see [26]). Fortunately, if $\dim Q = 2$, the solution exists and is unique, up to a single scalar parameter μ (see also [27]):

Proposition 2 *The invariant Euler-Poisson equation of the relativistic two-dimensional motion is:*

$$\begin{aligned} & -\frac{* \mathbf{v}''}{(1 + \mathbf{v} \cdot \mathbf{v})^{3/2}} + 3 \frac{* \mathbf{v}'}{(1 + \mathbf{v} \cdot \mathbf{v})^{5/2}} (\mathbf{v} \cdot \mathbf{v}') - \\ & - \frac{\mu}{(1 + \mathbf{v} \cdot \mathbf{v})^{3/2}} ((1 + \mathbf{v} \cdot \mathbf{v}) \mathbf{v}' - (\mathbf{v}' \cdot \mathbf{v}) \mathbf{v}) = \mathbf{0} \end{aligned} \quad (33)$$

The dual vector above is defined in commonly used notations, $(*\mathbf{w})_i = \epsilon_{ji} w^j$. We know two different Lagrange functions for the left hand side of (33):

$$L_1 = -\frac{\sqrt{2} v^1}{\sqrt{1 + v_i v^i} (1 + v_2 v^2)} + \mu \sqrt{1 + v_i v^i} \quad (34)$$

$$L_2 = \frac{\sqrt{1} v^2}{\sqrt{1 + v_i v^i} (1 + v_1 v^1)} + \mu \sqrt{1 + v_i v^i}. \quad (35)$$

These should differ by a total time derivative

$$L_2 - L_1 = \frac{d}{dt} F. \quad (36)$$

In fact, let $g_{ij} = \text{diag}(1, \eta_1, \eta_2)$, $\eta_i = \pm 1$. Then, if, for example, $\eta_1 \eta_2 = 1$, then

$$F = \arctan \frac{v^1 v^2}{\sqrt{1 + v_i v^i}}. \quad (37)$$

With the help of the prescriptions of Proposition 1 we immediately obtain the ‘homogeneous’ counterpart of (33):

$$\begin{aligned} & -\frac{\ddot{\mathbf{u}} \times \mathbf{u}}{\|\mathbf{u}\|^3} + 3 \frac{\dot{\mathbf{u}} \times \mathbf{u}}{\|\mathbf{u}\|^5} (\dot{\mathbf{u}} \cdot \mathbf{u}) - \\ & - \frac{\mu}{\|\mathbf{u}\|^3} ((\mathbf{u} \cdot \mathbf{u}) \dot{\mathbf{u}} - (\dot{\mathbf{u}} \cdot \mathbf{u}) \mathbf{u}) = \mathbf{0} \end{aligned} \quad (38)$$

with the corresponding family of Lagrange functions,

$$\begin{aligned} \mathcal{L}_1 &= \frac{u^1 (\dot{u}^0 u^2 - \dot{u}^2 u^0)}{\|\mathbf{u}\| (u^0 u^0 + u_2 u^2)} + \mu \|\mathbf{u}\|, \\ \mathcal{L}_2 &= \frac{u^2 (\dot{u}^1 u^0 - \dot{u}^0 u^1)}{\|\mathbf{u}\| (u_0 u^0 + u_1 u^1)} + \mu \|\mathbf{u}\|, \\ \mathcal{L}_0 &= \frac{u^0 (\dot{u}^2 u^1 - \dot{u}^1 u^2)}{\|\mathbf{u}\| (u_1 u^1 + u_2 u^2)} + \mu \|\mathbf{u}\|, \end{aligned}$$

where, for the sake of the ‘coordinate homogeneity’, the notation u^0 was introduced to substitute the evolution of the time coordinate t . One obtains the third expression for \mathcal{L}_0 by simple abuse of cyclic symmetry philosophy.

The difference between \mathcal{L}_2 and \mathcal{L}_1 is readily obtained from the Proposition 1 again. Thus in the case when (37) holds, one gets from (10) and (15)

$$\begin{aligned} \mathcal{L}_2 - \mathcal{L}_1 &= u^0 (L_2 \circ pr^2 - L_1 \circ pr^2) \\ &= \frac{d}{d\zeta} \arctan \frac{(v^1 \circ pr^2)(v^2 \circ pr^2)}{\sqrt{1 + (v_i \circ pr^2)(v^i \circ pr^2)}} \\ &= \frac{d}{d\zeta} \arctan \frac{u^1 u^2}{u_0 \sqrt{u_\alpha u^\alpha}}, \end{aligned}$$

and for two other differences by direct calculation and the trigonometric identity for arctan:

$$\begin{aligned} \mathcal{L}_1 - \mathcal{L}_0 &= \frac{d}{d\zeta} \arctan \frac{u^0 u^1}{u_2 \sqrt{u_\alpha u^\alpha}}, \\ \mathcal{L}_0 - \mathcal{L}_2 &= \frac{d}{d\zeta} \arctan \frac{u^0 u^2}{u_1 \sqrt{u_\alpha u^\alpha}}. \end{aligned}$$

To produce a variational equation of the third order, the Lagrange function should be of affine

type in second derivatives. It makes no sense to even try finding a Poincaré-invariant such Lagrange function in space-time dimensions greater than two [26]. But the generalized momentum

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}} - \frac{d}{d\zeta} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{u}}} = \frac{\dot{\mathbf{u}} \times \mathbf{u}}{\|\mathbf{u}\|^3} + \mu \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

does not depend on the particular choice of one of the above family of Lagrange functions. This expression for the generalized momentum was (in different notations) in fact obtained in [10] by means of introducing an abundance of Lagrange multipliers into the formulation of the corresponding variational problem.

4.1 Free relativistic top in two dimensions

This equation (38) carries certain amount of physical sense. We leave it to the reader to ensure (see also [28]) that in terms of spin vector (4) the Mathisson-Papapetrou equations (1-2) under the Mathisson-Pirani auxiliary condition (3) are equivalent to the next system of equations,

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma\delta} \ddot{u}^\beta u^\gamma \sigma^\delta - 3 \frac{\dot{\mathbf{u}} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \varepsilon_{\alpha\beta\gamma\delta} \dot{u}^\beta u^\gamma \sigma^\delta + \\ + \frac{m_0}{\sqrt{|g|}} [(\dot{\mathbf{u}} \cdot \mathbf{u}) u_\alpha - \|\mathbf{u}\|^2 \dot{u}_\alpha] = \mathcal{F}_\alpha \quad (39) \\ \|\mathbf{u}\|^2 \dot{\sigma}_\alpha + (\boldsymbol{\sigma} \cdot \dot{\mathbf{u}}) u_\alpha = 0 \\ \boldsymbol{\sigma} \cdot \mathbf{u} = 0. \end{aligned}$$

It should be clear that the four-vector $\boldsymbol{\sigma}$ is constant in all its components if the force \mathcal{F}_α vanishes. Equation (39) admits a planar motion, when $u_3 = \dot{u}_3 = \ddot{u}_3 = 0$, and, if we put $g_{\alpha\beta} = \text{diag}(1, \eta_1, \eta_2, \eta_3)$, it takes the shape of

$$\begin{aligned} \eta_3 \sigma_3 \left(\frac{\ddot{\mathbf{u}} \times \mathbf{u}}{\|\mathbf{u}\|^3} - 3 \frac{\dot{\mathbf{u}} \times \mathbf{u}}{\|\mathbf{u}\|^5} (\dot{\mathbf{u}} \cdot \mathbf{u}) \right) + \\ + \frac{m_0}{\|\mathbf{u}\|^3} [(\mathbf{u} \cdot \mathbf{u}) \dot{\mathbf{u}} - (\dot{\mathbf{u}} \cdot \mathbf{u}) \mathbf{u}] = \mathbf{0}, \end{aligned}$$

where the vector \mathbf{u} becomes three-dimensional. Comparing with (38) imposes $\mu = \frac{m_0}{\eta_3 \sigma_3}$.

5 Poisson structure

In constructing Hamilton equations we follow the prescriptions of [29]. First, let us introduce, along

with the variables x^i, v^i, v'^i , yet more variables, \mathbf{p}_i and \mathbf{r}_i . We define the following Hamilton function H on the total set of variables $x^i, v^i, v'^i, \mathbf{p}_i$ and \mathbf{r}_i ,

$$H = \mathbf{p} \cdot \mathbf{v} + \mathbf{r} \cdot \mathbf{v}' - L. \quad (40)$$

The system of higher-order equations of motion in these variables take the shape

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{v}}{dt} = \frac{\partial H}{\partial \mathbf{r}}, \quad (41)$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}, \quad (42)$$

$$\frac{d\mathbf{r}}{dt} = -\frac{\partial H}{\partial \mathbf{v}}, \quad \frac{\partial H}{\partial \mathbf{v}'} = \mathbf{0}. \quad (43)$$

Equations (41) in fact merely ensure that only holonomic enter into consideration:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \mathbf{v}'.$$

Equations (43) give the definition of Ostrogradsky momenta:¹

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} - \frac{d\mathbf{r}}{dt}, \quad \mathbf{r} = \frac{\partial L}{\partial \mathbf{v}'}. \quad (44)$$

Equation (42) is the variational equation of motion (the Euler-Poisson vector equation)

$$\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} - \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}'} \right) = \mathbf{0}.$$

As soon as any Lagrange function that corresponds to the equation (33), should be of affine type with respect to the accelerations, the definition (40) reduces to

$$H = \mathbf{p} \cdot \mathbf{v} - \mu \sqrt{1 + v_i v^i}. \quad (45)$$

Let us take the Lagrange function (34) in place of L . Then equation (43) in its first component produces

$$\frac{dr_1}{dt} = -\frac{\partial H}{\partial v^1}, \quad r_1 = 0. \quad (46)$$

Now the consistency condition for the pair of equations (46) reads

$$\frac{\partial H}{\partial v^1} = 0, \quad (47)$$

¹Although we use bold character to denote the two-component array \mathbf{p} , we have to warn the Reader that, whereas \mathbf{r} transforms like a vector, \mathbf{p} does not, quite the same way as neither does \mathbf{x} .

which, together with

$$\frac{\partial H}{\partial v'^2} = 0 \quad (48)$$

from (43), allows us to get rid not only of the variable v'^2 (v'^1 is already out of game by $\frac{\partial L_1}{\partial v'^1} = 0$), but also of v^1 , thus reducing the overall number of canonical variables by two (i.e. ignoring the conjugated pair (v^1, r_1)). The theory is non-degenerate in the sense that the Hessian

$$\begin{aligned} & \left\| \begin{array}{cc} \frac{\partial^2 L_1}{\partial v'^2 \partial v'^2} & \frac{\partial^2 L_1}{\partial v'^2 \partial v^1} \\ \frac{\partial^2 L_1}{\partial v^1 \partial v'^2} & \frac{\partial^2 L_1}{\partial v^1 \partial v^1} \end{array} \right\| \\ &= \left\| \begin{array}{cc} 0 & -\frac{1}{(1+v_i v^i)^{3/2}} \\ -\frac{1}{(1+v_i v^i)^{3/2}} & 3\frac{v'^2 v_1}{(1+v_i v^i)^{5/2}} + \mu \eta_1 \frac{1+v_2 v^2}{(1+v_i v^i)^{3/2}} \end{array} \right\| \end{aligned}$$

is of the rank 2.

Let us solve equation (48) for v^1 . With L_1 in place of L , equation (48) reads

$$r_2 = -\frac{v^1}{\sqrt{1+\mathbf{v}\cdot\mathbf{v}}(1+v_2 v^2)}. \quad (49)$$

Now take the square of both sides:

$$\eta_1 r_2 r_2 (1+\mathbf{v}\cdot\mathbf{v})(1+v_2 v^2)^2 = v_1 v^1,$$

from where by collecting like terms,

$$\begin{aligned} v_1 v^1 \left(1 - \eta_1 r_2 r_2 (1+v_2 v^2)^2 \right) \\ = \eta_1 r_2 r_2 (1+v_2 v^2)^3. \end{aligned} \quad (50)$$

Let us construct the expression $(1+\mathbf{v}\cdot\mathbf{v})$ from (50) as follows:

$$\begin{aligned} & (1+\mathbf{v}\cdot\mathbf{v}) \left(1 - \eta_1 r_2 r_2 (1+v_2 v^2)^2 \right) \\ &= 1 - \eta_1 r_2 r_2 (1+v_2 v^2)^2 \\ & \quad + \eta_1 r_2 r_2 (1+v_2 v^2)^3 + v_2 v^2 \\ & \quad - \eta_1 v_2 v^2 r_2 r_2 (1+v_2 v^2)^2 \\ &= 1 + v_2 v^2 + (1+v_2 v^2)^2 \cdot 0 \\ &= 1 + v_2 v^2. \end{aligned} \quad (51)$$

The expression for the variable v'^2 might have been obtained from (47), but actually we are not interested in it as far as we are going to use directly formula (45) rather than (40). By (49) and (51) the Hamiltonian (45) becomes

$$\begin{aligned} H &= p_2 v^2 - p_1 r_2 \frac{(1+v_2 v^2)^{3/2}}{\sqrt{1-\eta_1 r_2 r_2 (1+v_2 v^2)^2}} \\ & \quad - \mu \sqrt{\frac{1+v_2 v^2}{1-\eta_1 r_2 r_2 (1+v_2 v^2)^2}}. \end{aligned} \quad (52)$$

The Poisson structure is implemented by the Poisson bracket

$$\{F, G\}_{\mathbf{p}, \mathbf{r}} = \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} + \frac{\partial F}{\partial v^2} \frac{\partial G}{\partial r_2} - \frac{\partial F}{\partial r_2} \frac{\partial G}{\partial v^2},$$

and the generalized Hamilton equations read:

$$\begin{aligned} \frac{dx^i}{dt} &= \{x^i, H\}_{\mathbf{p}, \mathbf{r}} & \frac{dp_i}{dt} &= \{p_i, H\}_{\mathbf{p}, \mathbf{r}} \\ \frac{dv^2}{dt} &= \{v^2, H\}_{\mathbf{p}, \mathbf{r}} & \frac{dr_2}{dt} &= \{r_2, H\}_{\mathbf{p}, \mathbf{r}}. \end{aligned}$$

If we had started with (35), then definition (40) would have changed to new variables, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{r}}$, and a new function \tilde{H} ,

$$\tilde{H} = \tilde{\mathbf{p}}\cdot\mathbf{v} + \tilde{\mathbf{r}}\cdot\mathbf{v}' - L_2. \quad (53)$$

In place of (46), (47), and (48) we should have had

$$\frac{d\tilde{r}_2}{dt} = -\frac{\partial \tilde{H}}{\partial v^2}, \quad \tilde{r}_2 = 0, \quad (54)$$

$$\frac{\partial \tilde{H}}{\partial v^2} = 0.$$

and

$$\frac{\partial H}{\partial v^1} = 0 \quad (55)$$

from (43). Formulae (49) and (51) would have been substituted by

$$\tilde{r}_1 = \frac{v^2}{\sqrt{1+\mathbf{v}\cdot\mathbf{v}}(1+v_1 v^1)} \quad (56)$$

and

$$(1+\mathbf{v}\cdot\mathbf{v}) \left(1 - \eta_2 \tilde{r}_1 \tilde{r}_1 (1+v_1 v^1)^2 \right) = 1 + v_1 v^1. \quad (57)$$

The theory becomes non-degenerate with the Hessian

$$\left\| \begin{array}{cc} \frac{\partial^2 L_2}{\partial v'^1 \partial v'^1} & \frac{\partial^2 L_2}{\partial v'^1 \partial v^2} \\ \frac{\partial^2 L_2}{\partial v^2 \partial v'^1} & \frac{\partial^2 L_2}{\partial v^2 \partial v^2} \end{array} \right\| = \left\| \begin{array}{cc} 0 & \frac{1}{(1+v_i v^i)^{3/2}} \\ \frac{1}{(1+v_i v^i)^{3/2}} & -3 \frac{v'^1 v_2}{(1+v_i v^i)^{5/2}} + \mu \eta_2 \frac{1+v_1 v^1}{(1+v_i v^i)^{3/2}} \end{array} \right\|$$

With (56) and (57) the new Hamilton function (53) reads

$$\tilde{H} = \tilde{p}_1 v^1 + \tilde{p}_2 \tilde{r}_1 \frac{(1+v_1 v^1)^{3/2}}{\sqrt{1-\eta_2 \tilde{r}_1 \tilde{r}_1 (1+v_1 v^1)^2}} - \mu \sqrt{\frac{1+v_1 v^1}{1-\eta_2 \tilde{r}_1 \tilde{r}_1 (1+v_1 v^1)^2}}. \quad (58)$$

The new Poisson structure would have been given by

$$\{F, G\}_{\tilde{\mathbf{p}}, \tilde{\mathbf{r}}} = \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial \tilde{p}_i} - \frac{\partial F}{\partial \tilde{p}_i} \frac{\partial G}{\partial x^i} + \frac{\partial F}{\partial v^1} \frac{\partial G}{\partial \tilde{r}_1} - \frac{\partial F}{\partial \tilde{r}_1} \frac{\partial G}{\partial v^1}, \quad (59)$$

and the new generalized Hamilton equations would have read:

$$\begin{aligned} \frac{dx^i}{dt} &= \{x^i, H\}_{\tilde{\mathbf{p}}, \tilde{\mathbf{r}}} & \frac{dp_i}{dt} &= \{p_i, H\}_{\tilde{\mathbf{p}}, \tilde{\mathbf{r}}} \\ \frac{dv^1}{dt} &= \{v^1, H\}_{\tilde{\mathbf{p}}, \tilde{\mathbf{r}}} & \frac{dr_1}{dt} &= \{r_1, H\}_{\tilde{\mathbf{p}}, \tilde{\mathbf{r}}}. \end{aligned}$$

The Hamilton function in (40) is defined on the space $T^2 Q \times_{TQ} T^*(TQ)$ with coordinates $\mathbf{x}, \mathbf{v}, \mathbf{v}', \mathbf{p}, \mathbf{r}$ and may be lifted to the so-called *unified phase space* [30] $T^3 Q \times_{TQ} T^*(TQ)$ with coordinates $\mathbf{x}, \mathbf{v}, \mathbf{v}', \mathbf{v}'', \mathbf{p}, \mathbf{r}$ by the projection ignoring \mathbf{v}'' . The *Legendre–Ostrogradsky map* $T^3 Q \rightarrow T^*(TQ)$ over TQ (see again [30]), defined by formulæ (44), in our case of affine Lagrangian actually is defined on the space $T^2 Q$ and may be thought of as a graph in the space $T^2 Q \times_{TQ} T^*(TQ)$. It may be described by a pair of applications, $\mathbf{p} = \pi(\mathbf{v}, \mathbf{v}')$, and $\mathbf{r} = \rho(\mathbf{v}, \mathbf{v}')$. Given another Hamilton function, \tilde{H} ,

one arrives at another Legendre–Ostrogradsky map, $\tilde{\mathbf{p}} = \tilde{\pi}(\mathbf{v}, \mathbf{v}')$, and $\tilde{\mathbf{r}} = \tilde{\rho}(\mathbf{v}, \mathbf{v}')$. As far as (36) holds, and assuming $\frac{\partial F}{\partial \mathbf{v}'} = 0$, from (44) one gets

$$\tilde{\mathbf{r}} \stackrel{\text{def}}{=} \frac{\partial L_2}{\partial \mathbf{v}'} \equiv \frac{\partial L_1}{\partial \mathbf{v}'} + \frac{\partial}{\partial \mathbf{v}'} \left(\mathbf{v} \cdot \frac{\partial F}{\partial \mathbf{x}} + \mathbf{v}' \cdot \frac{\partial F}{\partial \mathbf{v}} \right) = \mathbf{r} + \phi, \quad (60)$$

$$\text{where } \phi \stackrel{\text{def}}{=} \frac{\partial F}{\partial \mathbf{v}}, \quad (61)$$

$$\begin{aligned} \tilde{\mathbf{p}} \stackrel{\text{def}}{=} \frac{\partial L_2}{\partial \mathbf{v}} - \frac{d\tilde{\mathbf{r}}}{dt} &= \mathbf{p} + \left(\frac{\partial}{\partial \mathbf{v}} - \frac{d}{dt} \frac{\partial}{\partial \mathbf{v}'} \right) \frac{dF}{dt} \\ &= \mathbf{p} + \frac{\partial F}{\partial \mathbf{x}}. \end{aligned}$$

In our example $\frac{\partial F}{\partial \mathbf{x}} = 0$, so one can drop the diacritic tilde over \mathbf{p} and π . From (60) one can obtain the expression for ϕ_1 , using (56) together with the second equation of (46), and the expression for ϕ_2 , using (49) accompanied by the second equation of (54):

$$\phi_1 = \tilde{r}_1 = \frac{v^2}{\sqrt{1+\mathbf{v} \cdot \mathbf{v}}(1+v_1 v^1)}, \quad (62.1)$$

$$\phi_2 = -r_2 = \frac{v^1}{\sqrt{1+\mathbf{v} \cdot \mathbf{v}}(1+v_2 v^2)}. \quad (62.2)$$

One can easily check

$$\frac{\partial \phi_1}{\partial v^2} = \frac{1}{(1+\mathbf{v} \cdot \mathbf{v})^{3/2}} = \frac{\partial \phi_2}{\partial v^1}, \quad (63)$$

which should be obvious from (61).

Let us consider the following diagram

$$\begin{array}{ccc} & \{\mathbf{v}, \mathbf{v}'\} & \\ (id, \pi, \rho) \swarrow & & \searrow (id, \pi, \tilde{\rho}) \\ \{\mathbf{v}, \mathbf{p}, \mathbf{r}\} & \xrightarrow{(id, id, \Phi)} & \{\mathbf{v}, \mathbf{p}, \tilde{\mathbf{r}}\} \\ H \searrow & & \swarrow \tilde{H} \\ & \mathbb{R} & \end{array} \quad (64)$$

where the map Φ is given by $\Phi(\mathbf{v}, \mathbf{p}, \mathbf{r}) = \mathbf{r} + \phi(\mathbf{v})$.

Proposition 3 *The diagram (64) commutes*

Proof. By the second formula in (46), together with (62.2), in the upper triangle to the left we

get

$$\rho_1(\mathbf{v}, \mathbf{v}') = 0; \quad \rho_2(\mathbf{v}, \mathbf{v}') = -\phi_2,$$

whereas by (62.1) and (54) to the right we have

$$\tilde{\rho}_1(\mathbf{v}, \mathbf{v}') = \phi_1; \quad \tilde{\rho}_2(\mathbf{v}, \mathbf{v}') = 0. \quad (65)$$

On the right by (62.1) one calculates

$$\begin{aligned} (\Phi \circ \rho)_1(\mathbf{v}, \mathbf{v}') &= \Phi_1(\rho_1(\mathbf{v}), \rho_2(\mathbf{v})) = 0 + \phi_1 = \phi_1, \\ (\Phi \circ \rho)_2(\mathbf{v}, \mathbf{v}') &= \Phi_2(\rho_1(\mathbf{v}), \rho_2(\mathbf{v})) = -\phi_2 + \phi_2 = 0, \end{aligned}$$

which coincides with (65).

In the lower triangle we compute the composition

$$\begin{aligned} \tilde{H} \circ (id, id, \Phi)(\mathbf{v}, \mathbf{p}, \mathbf{r}) &= \mathbf{p} \cdot \mathbf{v} + \Phi(\mathbf{v}, \mathbf{p}, \mathbf{r}) \cdot \mathbf{v}' - L_2 \\ &= \mathbf{p} \cdot \mathbf{v} + \mathbf{r} \cdot \mathbf{v}' + \phi \cdot \mathbf{v}' - L_2 = \mathbf{p} \cdot \mathbf{v} + \mathbf{r} \cdot \mathbf{v}' - L_1 \\ &= H(\mathbf{v}, \mathbf{p}, \mathbf{r}), \end{aligned}$$

because $\phi \cdot \mathbf{v}' = L_2 - L_1$ in view of (62).

Q.E.D.

Let us turn to the notion of canonical transformations. Consider some space of conjugated pairs of variables $\{Q^A, P_A\}$ with a Poisson bracket

$$\{F, G\}_{P,Q} = \frac{\partial F}{\partial Q^A} \frac{\partial G}{\partial P_A} - \frac{\partial F}{\partial P_A} \frac{\partial G}{\partial Q^A}$$

and a transformation

$$(Q^A, P_A) \mapsto (\tilde{Q}^A, \tilde{P}_A). \quad (66)$$

Definition 1 Transformation (66) is called canonical if the following two equivalent conditions hold [2, 29]:

1. The Jacobian matrix of the inverse transformation to (66) reads

$$\left\| \begin{pmatrix} \left(\frac{\partial \tilde{P}}{\partial \mathbf{P}} \right)^T & - \left(\frac{\partial \tilde{Q}}{\partial \mathbf{P}} \right)^T \\ - \left(\frac{\partial \tilde{P}}{\partial \mathbf{Q}} \right)^T & \left(\frac{\partial \tilde{Q}}{\partial \mathbf{Q}} \right)^T \end{pmatrix} \right\| \quad (67)$$

where superscript T denotes the transposition of the embraced matrix.

2. The Poisson bracket of the new conjugate pairs calculated with respect to the old variables does not change:

$$\begin{aligned} \{\tilde{Q}^A, \tilde{Q}^B\}_{P,Q} &= 0; \quad \{\tilde{P}_A, \tilde{P}_B\}_{P,Q} = 0; \\ \{\tilde{Q}^A, \tilde{P}_B\}_{P,Q} &= \delta_B^A. \end{aligned} \quad (68)$$

Proposition 4 Let $\eta : \{\mathbf{x}, \mathbf{v}, \mathbf{p}, \mathbf{r}\} \rightarrow \{\mathbf{x}, \mathbf{v}, \mathbf{p}, \tilde{\mathbf{r}}\}$ be defined as $\eta = (id, id, id, \Phi)$ with Φ from the diagram (64). Then η is canonical.

Proof. We shall prove both properties in Definition (1).

Proof of property (1). The Jacobian matrix of η is

$$\left\| \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial \phi_1}{\partial \mathbf{v}^1} & \frac{\partial \phi_1}{\partial \mathbf{v}^2} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \phi_2}{\partial \mathbf{v}^1} & \frac{\partial \phi_2}{\partial \mathbf{v}^1} & 0 & 0 & 0 & 1 \end{array} \right\| \quad (69)$$

Obviously its determinant is 1.

The inverse to (69) is

$$\left\| \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\partial \phi_1}{\partial \mathbf{v}^1} & -\frac{\partial \phi_1}{\partial \mathbf{v}^2} & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\partial \phi_2}{\partial \mathbf{v}^1} & -\frac{\partial \phi_2}{\partial \mathbf{v}^1} & 0 & 0 & 0 & 1 \end{array} \right\| \quad (70)$$

In order to apply (67) to (70) we put $Q^A = (x^i, v^i)$ and $P_A = (p_i, r_i)$ and then we profit from (63) to check that (67) holds.

Q.E.D.

Proof of property (2). By the definition of η we are led to put $\tilde{Q}^A = Q^A$, $\tilde{\mathbf{p}}_i = \mathbf{p}_i$, and $\tilde{\mathbf{r}}_i = \mathbf{r}_i + \phi_i$, so that

$$\begin{aligned}\{\tilde{Q}^A, \tilde{Q}^B\} &= \{Q^A, Q^B\} = 0; \\ \{\tilde{\mathbf{v}}^i, \tilde{\mathbf{p}}_j\} &= \{\mathbf{v}^i, \mathbf{p}_j\} = 0; \quad \{\tilde{\mathbf{p}}_i, \tilde{\mathbf{p}}_j\} = \{\mathbf{p}_i, \mathbf{p}_j\} = 0; \\ \{\tilde{\mathbf{x}}^i, \tilde{\mathbf{r}}_j\} &= \{\mathbf{x}^i, \mathbf{r}_j + \phi_j\} = \{\mathbf{x}^i, \phi_j\} = 0; \\ \{\tilde{\mathbf{p}}_i, \tilde{\mathbf{r}}_j\} &= \{\mathbf{p}_i, \mathbf{r}_j + \phi_j\} = \{\mathbf{p}_i, \phi_j\} = 0; \\ \{\tilde{\mathbf{x}}^i, \tilde{\mathbf{p}}_j\} &= \{\mathbf{x}^i, \mathbf{p}_j\} = \delta_j^i; \\ \{\tilde{\mathbf{v}}^i, \tilde{\mathbf{r}}_j\} &= \{\mathbf{v}^i, \mathbf{r}_j + \phi_j\} = \{\mathbf{v}^i, \mathbf{r}_j\} = \delta_j^i \\ \{\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j\} &= \{\mathbf{r}_i + \phi_i, \mathbf{r}_j + \phi_j\} = \{\mathbf{r}_i, \phi_j\} + \{\phi_i, \mathbf{r}_j\} \\ &= -\frac{\partial \phi_j}{\partial \mathbf{v}^k} \delta_i^k + \frac{\partial \phi_i}{\partial \mathbf{v}^k} \delta_j^k = 0\end{aligned}$$

on the strength of (63).

This agrees with (68).

Q.E.D.

References

- [1] Krupková O. The geometry of ordinary variational equations. – Berlin e.a.: Springer, 1997. (↑ 341), (↑ 346)
- [2] De Leon M., Rodrigues Paulo R. Generalized classical mechanics and field theory. – Amsterdam e.a.: North Holland, 1985. (↑ 341), (↑ 351)
- [3] Saunders D. J. The geometry of jet bundles. – Cambridge e.a.: Cambridge Univ. Press, 1989. (↑ 341)
- [4] Mathisson M. Neue Mechanik materieller Systeme. // Acta Phys. Polon. – 1937. – **6**, fasc. III. – P. 163–200. (↑ 341)
- [5] Bopp F. Feldmechanische Begründung der Diracschen Wellengleichung. // Zf. für Naturf. – 1948. – **3a**, Ht. 8–11. – P. 564–573. (↑ 341)
- [6] Weyssenhoff J., Raabe A. Relativistic dynamics of spin-fluids and spin-particles. // Acta Phys. Polon. – 1947. – **9**, fasc. I. – P. 7–18. (↑ 341)
- [7] Hönl H. Mechanik und Massenspektrum der Elementarteilchen. // Zf. für Naturf. – 1948, **3a** Ht. 8–11. – P. 573–583. (↑ 341)
- [8] Tulczyjew W. Motion of multipole particles in general relativity theory. // Acta. Phys. Polon. – 1959. – **18**, fasc. V. – P. 393–409. (↑ 341)
- [9] Riewe F. Relativistic classical spinning-particle mechanics. // Il Nuovo Cim. – 1972. – **8B**, N. 1. – P. 271–277. (↑ 341)
- [10] Plyushchay M. S. Relativistic massive particle with higher curvatures as a model for the description of bosons and fermions. // Phys. Lett. B. – 1990. – **235**, No. 1. – P. 47–51. (↑ 341), (↑ 348)
- [11] Nesterenko V. V., Feoli A. and Scarpetta G. Dynamics of relativistic particles with Lagrangians dependent on acceleration. // J. Math. Phys. – 1995. – **36**, No. 10. – P. 5552–5564. (↑ 341)
- [12] Лейко С. Г. Экстремали функционалов поворота для кривых псевдориманова пространства и траектории спин-частиц в гравитационных полях. // ДАН. – 1992. – **325**, № 4. – С. 659–663. (↑ 341)
English translation: Leiko S. G. Extremals of rotation functionals of curves in a pseudo-Riemannian space, and trajectories of spinning particles in gravitational fields. // Dokl. Math. – 1993. – **46**, No. 1. – P. 84–87.
- [13] Arodź H., Sitarz A. and Węgrzyn P. On relativistic point particles with curvature-dependent actions. // Acta Phys. Polon. – 1989. – **B20**, fasc. XI. – P. 921–939. (↑ 341)
- [14] Нерсисян А. П. Лагранжева модель безмассовой частицы на пространственноподобных кривых. // Теор. Мат. Физика. – 2001. – **126**, N° 2. – С. 179–195. (↑ 341)
[Nersesyan A. P. // Teor. Mat. Fizika. – 2001. – **126**, No. 2. – P. 179–195 (in Russian)]
English translation: Nersesyan A. P. Lagrangian model of a massless particle on spacelike curves Theoretical and Mathematical Physics. // Theoretical and Mathematical Physics. – 2001. **126**, No. 2. – P. 147–160.
- [15] Arreaga G., Capovilla R. and Guven J. Frenet–Serret dynamics. // Class. Quant. Grav. – 2001. – **18**, No. 23. – P. 5065–5083. (↑ 341)
- [16] Scarpetta G. Relativistic kinematics with Caianiello’s maximal proper acceleration. // Lett. Nuovo. Cim. – 1984. – **41**, N. 2. – P. 51–58. (↑ 341)
- [17] Plyatsko R. Gravitational ultrarelativistic spin-orbit interaction and the weak equivalence principle. // Phys. Rev. D. – 1998. – **58**. – P. 084031–1–5. (↑ 341)

- [18] Пляцко Р. М. // Прояви гравітаційної ультра-релятивістської спин-орбітальної взаємодії. – Київ: Наукова думка, 1988. [Plyatsko R. M. // Manifestations of gravitational ultra-relativistic spin-orbital interaction. – Kyiv: Naukova Dumka, 1988 (in Ukrainian)] (↑ [342](#))
- [19] Якупов М. Ш. Редукция уравнений движения пробной частицы со спином. // Гравитация и теория относительности. – Казань: Изд-во Каз. ун-та, 1983, вып. 19. – С. 146–162. <MR0721859 (85b:83007)> (↑ [342](#))
[Yakupov M. Sh. Reduction of the equations of motion of a test particle with spin. // Gravitation and Theory of Relativity. – Kazan': Kazan' Univ. Press, 1983, issue 19. – P. 146–162 (in Russian)]
- [20] Matsyuk R. Ya. Autoparallel variational description of the free relativistic top third order dynamics // *In book: Differential Geometry and Applications. Proc. 8th conf. DGA. Opava, August 27–31 2001.* – Opava: Silesian Univ, 2001. – P. 447–459. [ArXiv:1407.3371v1](#) <MR1978798 (2004d:70025)> (↑ [344](#))
- [21] Tulczyjev W. Sur la différentielle de Lagrange. // C. R. Acad. Sci. Paris. Sér. A et B. – 1975. – **280**, N° 19. – P. 1295–1298. (↑ [345](#))
- [22] Kolář I. On the Euler–Lagrange differential in fibered manifolds. // Rep. Math. Phys. – 1977. – **12**, No. 3. – P. 301–305. (↑ [346](#))
- [23] Matsyuk R. Ya. Integration by parts and vector differential forms in higher order variational calculus on fibred manifolds. // *Matematychni Studii [Matematchni Studii]*. – 1999. – **11**, No. 1. – P. 85–107. [arXiv:1406.3369v2](#) <MR1686048 (2000b:58031)> (↑ [346](#))
- [24] Мацюк Р. Я. О существовании лагранжиана для неавтономной системы обыкновенных дифференциальных уравнений. // *Мат. Методы и физ.-мех. поля. Вып. 20.* – Киев: Наукова думка, 1984. – С. 16–19. [Full text: R^G](#) <MR0756973 (85g:70020)> (↑ [346](#))
[Matsyuk R. Ya. Existence of a Lagrangian for a nonautonomous system of ordinary differential equations. // *Mat. Metody i Fiz.-Mekh. Polya. Issue 20.* – Kiev.: Nauk. Dumka, 1984. – P. 16–19 (in Russian)]
- [25] Matsyuk R. Ya. Symmetries of vector exterior differential systems and the inverse problem in second-order Ostrograds'kii mechanics. // J. Nonlinear Math. Phys. – 1997. – **4**, No. 1–2. – P. 89–97. [ArXiv:1406.5877v1](#) <MR1401574 (97h:58009)> (↑ [347](#))
- [26] Мацюк Р. Я. Пуанкаре-инвариантные уравнения движения в лагранжевой механике с высшими производными. Дис. ... к-та физ.-мат. наук. – Львов, 1984. (↑ [347](#)), (↑ [348](#))
[Matsyuk R. Ya. Poincaré-invariant equations of motion in Lagrangean mechanics with higher derivatives. Thesis. Lviv, 1984 (in Russian)]
- [27] Matsyuk R. Ya. Third-order relativistic dynamics: classical spinning particle travelling in a plane. // *Condensed Matter Phys.* – 1998. – **1**, No. 3(15). – P. 453–462. [arXiv:1304.7494v1](#) (↑ [347](#))
- [28] Мацюк Р. Я. Варіаційне узагальнення вільної релятивістської дзиги. // *Фізичний збірник НТШ.* – 2006. – **6**. – С. 206–214. [arXiv:1407.7009v1](#) (↑ [348](#))
[Matsyuk R. Ya. Variational generalization of free relativistic top. // *Fizycznyj zbirnyk NTSh.* – 2006. – **6**. – P. 206–214 (in Ukrainian)]
- [29] Гитман Д. М., Тютин И. В. Каноническое квантование полей со связями. – М.: Наука, 1986. (↑ [348](#)), (↑ [351](#))
English translation: Gitman D. M., Tyutin I. V. Quantization of fields with constraints. – Berlin: Springer-Verlag, 1990.
- [30] Prieto-Martinez P. D, Román-Roy N. Lagrangian–Hamiltonian unified formalism for autonomous higher order dynamical systems. // J. Phys. A: Math. Theor. – 2011. – **44**, No. 38. – 385203. (↑ [350](#))